International Journal of Theoretical Physics, Vol. 15, No. 11 (1976), pp. 877–884

On the "Near to Minimal" Canonical Realizations of the Lie Algebra C_n

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Received: 20 *October* 1975

Abstract

It is proved that canonical realizations of the Lie algebra C_n in the quotient division ring $D_{2(2n-2)}$ of the Weyl algebra $W_{2(2n-2)}$ in $2n - 2$ quantum canonical pairs are, in a definite sense, related to the standard minimal one in $D_{2n} \subset D_{2(2n-2)}$. Further, in any realization of C_n in $W_{2(2n-1)}$ all Casimir operators are realized by multiples of identity element.

1. *Introduction*

The main purpose of this note is to prove two assertions concerning the canonical realizations of the Lie algebra $C_n \simeq sp(2n, C)$, $n > 1$, through rational functions or polynomials in a certain number of quantum canonical pairs p_i and q_i. More precisely we are interested in realizations of C_n in the Wey algebra $W_{2(2n-1)}$, i.e., through polynomials in $2n - 1$ canonical pairs or in the quotient division ring $D_{2(2n-2)}$ of $W_{2(2n-2)}$, i.e., through rational functions in $2n - 2$ canonical pairs.

It is well known (Simoni and Zaccaria, 1969; Joseph, 1974) that canonical realizations of the Lie algebra C_n do not exist in D_{2m} if $m < n$. If $m = n$, all realizations of C_n are related through an endomorphism of D_{2n} ("equivalent") to one standard realization τ_0 and Casimir operators are realized by multiples of identity element (we call such realizations Schur realizations).

In this note we generalize the concept of related realizations and derive, first, a sufficient condition for realizations of C_n in D_{2m} with any $m \geq n$ to be related to the standard realization τ_0 in $D_{2n} \subset D_{2m}$. In combination with Joseph's result (Joseph, 1974) it gives our first result: Any realization of C_n in $D_{2(2n-2)}$ is related to τ_0 in $D_{2n} \subset D_{2(2n-2)}$. As a consequence, the value of every Casimir operator in any realization in $D_{2(2n-2)}$ is the same as in realization τ_0, which particularly means that all realizations of C_n in $D_{2(2n-2)}$ are Schur realizations. Our second result is that this last property is conserved in $W_{2(2n-1)}$, though there appear new realizations that are not related to realization τ_0 (Havlíček and Lassner, 1975).

In the Conclusions we compare C_n, in this respect, with the remaining complex classical Lie algebras.

2. *Preliminaries*

A. In C_n of Cartan's classification of complex simple Lie algebras we choose a basis of $n(2n + 1)$ elements $X_\beta{}^\alpha = -\epsilon_\alpha \epsilon_\beta X_{-\alpha}^{-\beta}$ satisfying

$$[X_\beta{}^\alpha, X_\delta{}^\gamma] = \delta_\beta{}^\gamma X_\delta{}^\alpha - \delta_\delta{}^\alpha X_\beta{}^\gamma + \epsilon_\alpha \epsilon_\beta \delta_\delta{}^{-\beta} X_{-\alpha}^\gamma + \epsilon_\beta \epsilon_\gamma \delta_{-\alpha}^\gamma X_\delta{}^{-\beta} \tag{2.1}$$

$$\epsilon_\alpha = \operatorname{sign} \alpha \qquad \alpha, \beta, \gamma, \delta = \pm 1, \pm 2, \ldots, \pm n$$

B. A canonical realization of a Lie algebra (associated algebra, quotient division ring) G is a homomorphism of G in the Weyl algebra W_{2N} (i.e., in the complex algebra of polynomials in $2N$ variables q_α, p^β satisfying $[q_\alpha, q_\beta] = [p^\alpha, p^\beta] = 0$ $[q_\alpha, p^\beta] = -\delta_\alpha{}^\beta$ or in the associated quotient division ring D_{2N} (i.e., in the rational functions in q_α, p^β). For an exact definition and foundation of D_{2N} see Gelfand and Kirrilov (1966).

A canonical realization of a Lie algebra is called a Schur realization if any Casimir operator (i.e., any element from the center of the enveloping algebra) is realized by multiple of the identity.

Definition 1. Let realization

$$\tau: G \to D_{2n} \subset D_{2n'}$$

of the Lie algebra G in the quotient division subring D_{2n} of $D_{2n'}$ and realization

$$\tau': G \to D_{2n'}$$

be given. τ' is called related to τ i.f.f. a realization

$$\vartheta: D_{2n} \to D_{2n'}$$

exists such that

$$\vartheta \circ \tau = \tau'$$

This definition generalizes the concept of related realizations introduced in our previous papers. If $D_{2n} = D_{2n'}$ we obtain the old definition of related realizations in the sense that τ and τ' are said to be related iff either τ is related to τ' or τ' is related to τ.

3. *Properties of Realizations of C_n in $D_{2(2n-2)}$ and $W_{2(2n-1)}$*

Lemma 1. (i) The following formulas give a canonical realization τ_0 of the Lie algebra C_n in D_{2n}:

$$
\begin{aligned}
\tau_0(X_j{}^i) &= -q_jp^i - \tfrac{1}{2}\delta_j{}^i\mathbb{1}, && i,j>0\\
&= \quad q_0p^ip^{-j}, && i>0, j<0\\
&= -q_0^{-1}q_{-i}q_j, && i<0, j>0\\
\tau_0(X_n{}^i) &= -q_0p^i && i>0\\
&= -q_{-i} && i<0\\
\tau_0(X_i{}^n) &= \quad q_0^{-1}q_i(\tau_0(X_n{}^n)+\tfrac{1}{2}) && i>0\\
&= -(\tau_0(X_n{}^n)-\tfrac{1}{2})\,p_{-i} && i<0\\
\tau_0(X_n^{-n}) &= -q_0\\
\tau_0(X_{-n}^n) &= \quad q_0^{-1}(\tau_0(X_n{}^n)+\tfrac{3}{2})(\tau_0(X_n{}^n)+\tfrac{1}{2})\\
\tau_0(X_n{}^n) &= -q_0p^0 - q\cdot p
\end{aligned}
\tag{3.1}
$$

where $q\cdot p = q_0p^0 + \cdots + q_{n-1}p^{n-1}$ and $i,j = -(n-1), \cdots, -1, 1, \ldots, n-1$.

(ii) τ_0 is a Schur realization.

It is straightforward to verify that the generators $\tau_0(X_\beta{}^\alpha)$ from (3.1) obey the commutation relations (2.1) of C_n. The realization (3.1) is a minimal one since only n canonical pairs occur and therefore it must be a Schur realization (see Joseph, 1974).

If $n' \geqslant n$, the realization τ_0 can be defined in $D_{2n'}$. We then without mention assume $\tau_0(C_n) \subset D_{2n}(q_0, p^0, \ldots, q_{n-1}, p^{n-1}) \subset D_{2n'}$ where $D_{2n}(q_0, p^0, \ldots, q_{n-1}, p^{n-1})$ denotes the quotient division subring of $D_{2n'}$ generated by the first n canonical pairs $q_0, p^0, \ldots, q_{n-1}, p^{n-1}$. The realization τ_0 will be called the *standard minimal* one.

Lemma 2. Let τ be any nontrivial realization of C_n with $n \geqslant 2$ in $D_{2n'}$, $n' \geqslant n$, fulfilling the condition

$$\tau\{(X_l{}^k + \tfrac{1}{2}\delta_l{}^k)X_n^{-n} - X_n^{-l}X_n{}^k\} = 0 \tag{3.2}$$

at least for one positive pair $k, l = 1, 2, \ldots, n-1$. Then τ is related to τ_0, i.e., $\vartheta\circ\tau_0 = \tau$. The realization $\vartheta\colon D_{2n} \to D_{2n'}$ is defined by the relations

$$
\begin{aligned}
\vartheta(q_k) &= Q_k, & \vartheta(p^k) &= P^k\\
\vartheta(q_0) &= -\tau(X_n^{-n}), & \vartheta(p^0) &= [\tau(2X_n^{-n})]^{-1}[\tau(X_n{}^n) + Q_kP^k]\\
Q_k &= -\tau(X_n^{-k}), & P^k &= [\tau(X_n^{-n})]^{-1}\tau(X_n{}^k)
\end{aligned}
\tag{3.3}
$$

Proof. Let us for abbreviation write $\hat{X}_\beta{}^\alpha = \tau(X_\beta{}^\alpha)$. Note that, owing to the simplicity of the Lie algebra C_n, $\hat{X}_n^{-n} = 0$ implies $\hat{X}_\beta{}^\alpha = 0$ for all α, β, which is in contradiction with the assumption of nontriviality of τ. Therefore $\hat{X}_n^{-n}$ is a nonzero element of $D_{2n'}$ and we can define

$$P^k = (\hat{X}_n^{-n})^{-1}\hat{X}_n{}^k, \qquad Q_k = -\hat{X}_n^{-k} \qquad k = 1, \ldots, n-1$$
$$P^0 = (2\hat{X}_n^{-n})^{-1}(\hat{X}_n{}^n + Q_kP^k), \qquad Q_0 = -\hat{X}_n^{-n} \tag{3.4}$$

(summation over k)

It is easy to prove, using commutation relations (2.1), that they commute as n canonical pairs, i.e.,

$$[P^\alpha, Q_\beta] = \delta_\beta{}^\alpha, \qquad [Q_\alpha, Q_\beta] = [P^\alpha, P^\beta] = 0, \qquad \alpha, \beta = 0, 1, 2, \ldots, n-1$$

Further, one can show that the rational functions

$$\hat{Y}_l{}^k = \hat{X}_l{}^k + Q_lP^k + \tfrac{1}{2}\delta_l{}^k \qquad k, l = 1, \ldots, n-1 \tag{3.5}$$

$$\hat{Y}_l^{-k} = \hat{X}_l^{-k} + Q_0^{-1}Q_kQ_l \tag{3.6}$$

$$\hat{Y}^k_{-l} = \hat{X}^k_{-l} - Q_0P^kP^l \tag{3.7}$$

commute as the generators of C_{n-1}.

Condition (3.2) is equivalent to $\hat{Y}_l{}^k = 0$ and since C_{n-1} is a simple Lie algebra it follows that

$$\hat{Y}^k_{-l} = \hat{Y}_l^{-k} = \hat{Y}_l{}^k = 0 \text{ for all } k, l = 1, \ldots, n-1 \tag{3.8}$$

Thus from (3.5)-(3.7) we get

$$\begin{aligned} \hat{X}_j{}^i &= -Q_jP^i - \tfrac{1}{2}\delta_j{}^i, && i, j > 0 \\ &= Q_0P^iP^{-j}, && i > 0, j < 0 \\ &= -Q_0^{-1}Q_{-i}Q_j, && i < 0, j > 0 \end{aligned} \tag{3.9}$$

Further, (3.4) yields

$$\begin{aligned} \hat{X}_n{}^i &= -Q_0P^i, && i > 0 \\ &= -Q_{-i}, && i < 0 \end{aligned} \tag{3.10}$$

$$\hat{X}_n{}^n = -Q_0P^0 - Q \cdot P \tag{3.11}$$

where $Q \cdot P = Q_0P^0 + \cdots + Q_{n-1}P^{n-1}$.

Now we show that condition (3.8) implies

$$\hat{X}^n_{-k} = -(X_n{}^n - \tfrac{1}{2})P^k \qquad k = 1, 2, \ldots, n-1 \tag{3.12}$$

Consider condition (3.8) $Y^k_{-l} = 0$ especially for $k = l$.

Then $\hat{Y}^k_{-k} = 0$ gives, owing to (3.7) and (3.4),

$$\hat{X}_n^{-n}\hat{X}^k_{-k} = -\hat{X}_n{}^k\hat{X}_n{}^k \qquad k > 0 \text{ (no summation!)} \tag{3.13}$$

when the commutator $[X^n_{-n}, .]$ acts on this equation one gets

$$2\hat{X}_n{}^n\hat{X}^k_{-k} = \hat{X}^n_{-k}\hat{X}_n{}^k + \hat{X}_n{}^k\hat{X}^n_{-k}$$

and rewriting the right-hand side we obtain further

$$2\hat{X}_n{}^n\hat{X}^k_{-k} = 2\hat{X}^n_{-k}\hat{X}_n{}^k + [\hat{X}_n{}^k, \hat{X}^n_{-k}] = 2\hat{X}^n_{-k}\hat{X}_n{}^k + \hat{X}^k_{-k}$$

which implies

$$\hat{X}^n_{-k} = (\hat{X}_n{}^n - \tfrac{1}{2})(\hat{X}^k_{-k})(\hat{X}_n{}^k)^{-1}$$

(Note that $\hat{X}_n{}^k \neq 0$ for the same reasons as $\hat{X}_n^{-n} \neq 0$.) Thus, if we substitute $\hat{X}^k_{-k}$ by (3.9) and $\hat{X}_n{}^k$ by (3.10) we find the desired expression (3.12). The generators (3.9)-(3.12) are a generating set in the Lie algebra C_n, i.e., the remaining generators can be computed as commutators of these.

It follows from the commutation relation of P^α and Q_β that the mapping ϑ

$$\begin{aligned} \vartheta(q_\alpha) &= Q_\alpha \\ \vartheta(p^\alpha) &= P^\alpha \end{aligned} \qquad \alpha = 0, 1, \ldots, n-1$$

defines a realization of $D_{2n} = D_{2n}(q_0, p^0, \ldots, q_{n-1}, p^{n-1})$ in $D_{2n'}$. A comparision with (3.1) shows that $\vartheta \circ \tau_0 = \tau$ holds for the generating set of generators of C_n (3.9)-(3.12) and therefore this must be true for the whole algebra C_n.

Corollary 1. If for a nontrivial realization τ of C_n, $n \geqslant 2$, in $D_{2n'}$, $n' \geqslant n$, condition (3.2) is fulfilled, then for any Casimir operator z of C_n

$$\tau(z) = \tau_0(z) = \alpha_z \mathbb{1}, \alpha_z \in \mathbb{C}$$

i.e., τ is a Schur realization and the realization of any Casimir operator of C_n has the same eigenvalues as in the standard minimal realization τ_0.

Proof. The realization τ_0 is a Schur realization, i.e., $\tau_0(z) = \alpha_z \mathbb{1}$. The relation $\tau = \vartheta \circ \tau_0$ gives

$$\tau(z) = \vartheta[\tau_0(z)] = \vartheta(\alpha_z \mathbb{1}) = \alpha_z \vartheta(\mathbb{1}) = \alpha_z \mathbb{1}$$

because $\vartheta(\mathbb{1}) = \mathbb{1}$.

Theorem 1. Let τ be any realization of C_n, $n \geqslant 2$, in $D_{2n'}$, $n \leqslant n' \leqslant 2n-2$. Then τ is related to τ_0, i.e. $\tau = \vartheta \circ \tau_0$, where ϑ is given by equation (3.3). τ is a Schur realization and any Casimir operator has the same eigenvalues as in the standard minimal realization τ_0.

Proof. Theorem 1 follows from Lemma 2 if we show that for any realization τ of C_n in $D_{2n'}$, $n \leqslant n' \leqslant 2n-2$ the condition (3.2) is fulfilled. To show this, assume, on the contrary, that (3.2) does not hold. As equation

(3.2) is equivalent to equations (3.8), it means that all the generators (3.5)-(3.7) of the simple Lie algebra C_{n-1} are different from zero. Thus according to (3.4) we can define $n-1$ new canonical pairs

$$\begin{aligned} \tilde{P}^r &= [\hat{Y}_{n-1}^{-(n-1)}]^{-1}\hat{Y}_{n-1}^r, \qquad \tilde{Q}_r = -\hat{Y}_{n-1}^{-r} \\ \tilde{P}^0 &= [2\hat{Y}_{n-1}^{-(n-1)}]^{-1}(\hat{Y}_{n-1}^{n-1} + \tilde{Q}_r\tilde{P}^r), \qquad \tilde{Q}_0 = -\hat{Y}_{n-1}^{-(n-1)} \\ & \qquad r = 1, 2, \ldots, n-2 \end{aligned} \tag{3.14}$$

Since it can be verified that the $\hat{Y}_j^i$ and therefore the canonical pairs $\tilde{Q}_\rho$, P^σ commute with all Q_α, P^β defined by equations (3.4), we would have in $D_{2n'}$, $n \leqslant n' \leqslant 2n-2$, $2n-1$ canonical pairs. But this is impossible as in $D_{2n'}$ there do not exist more than n' canonical pairs (see Joseph, 1974). Therefore condition (3.2) must be fulfilled and we use Lemma 2 and Corollary 1.

Now we enlarge the number of canonical pairs to $2n-1$ and restrict ourselves to the Weyl algebra $W_{2(2n-1)}$.

Theorem 2. Any realization τ of the Lie algebra C_n in the Weyl algebra $W_{2(2n-1)}$ is a Schur realization.

Proof. For $n = 1$ the realization τ is minimal and therefore a Schur realization (Joseph, 1974). For $n \geqslant 2$ we first choose $2n-1$ commuting elements from the realization $\tau(UC_n) \subset W_{2(2n-1)}$ of the enveloping algebra UC_n. In notation from Lemma 2 and Theorem 1 they are

$$Q_0, Q_1, \ldots, Q_{n-1}$$

[see equation (3.4)] and

$$Q_0\tilde{Q}_0, Q_0\tilde{Q}_1, \ldots, Q_0\tilde{Q}_{n-2}$$

[see equations (3.14) and (3.6)]. Adding realization $\tau(z) = Z$ of any Casimir operator z, we obtain $2n$ commuting elements from $W_{2(2n-1)}$. In accordance with Joseph's result (Joseph, 1972, Theorem 3.3) only two possibilities can arise: (a) Either some of the $2n$ elements considered are realized by multiple of identity element; or, (b) if (a) does not hold, a (finite) set of nonzero complex numbers $\{\alpha_{ikl}\} \subset \mathbb{C}$ exists such that

$$\sum_{ikl} \alpha_{ikl} Q^i (Q_0\tilde{Q})^k Z^l = 0 \tag{3.15}$$

Here the multi index notation is used, i.e.

$$\alpha_{ikl} Q^i (Q_0\tilde{Q})^k Z^l = \alpha_{i_0,\ldots,i_{n-1},k_0,\ldots,k_{n-2},l} Q_0^{i_0} \cdots Q_{n-1}^{i_{n-1}} (Q_0 Q_0)^{k_0} \cdots (Q_0\tilde{Q}_{n-2})^{k_{n-2}} Z^l$$

We exclude the second possibility. For this we consider $W_{2(2n-1)}$ embedded in its quotient division ring $D_{2(2n-1)}$, where canonically conjugate variables $P^0, \ldots, P^{n-1}, \tilde{P}^0, \ldots, \tilde{P}^{n-2}$ exist [see equations (3.4), (3.14), (3.5)-(3.7)].

By means of multiple commutation of the variables P^α and $\tilde{P}^\rho$ with equation (3.15) we easily obtain

$$\sum_l \alpha_{ikl} Z^l = 0$$

for all i and k considered. A nontrivial polynomial

$$p_{ik}(Z) = \sum_l \alpha_{ikl} Z^l, \qquad \alpha_{ikl} \neq 0$$

in one variable Z can be written as the product

$$p_{ik}(Z) = \alpha_{ik} \prod_r (Z - \beta_{ik}^r \mathbb{1})^{n_r} = 0, \qquad \alpha_{ik}, \beta_{ik}^r \in \mathbb{C}$$

from which, as $W_{2(2n-1)}$ does not contain a nontrivial divisor of zero, we obtain

$$Z = \beta_{ik}^r \mathbb{1}$$

for some r. This contradicts, however, the assumption that (a) is not valid. So, the possibility (b) is excluded and we discuss the possibility (a). If some of the elements $Q_0 = -\hat{X}_n^{-n}, \ldots, Q_{n-1} = -\hat{X}_n^{-(n-1)}$ are multiples of identity then commutation relations (2.1) give immediately that such Q are equal to zero. It implies, owing to simplicity of the Lie algebra C_n, that all generators $\hat{X}_\beta{}^\alpha$ are zero, i.e., the realization is trivially a Schur realization. If some $Q_0\tilde{Q}_0, \ldots, Q_0\tilde{Q}_{n-2}$ is a multiple of identity, i.e., if

$$\hat{X}_n^{-n} \hat{Y}_{n-1}^{-k} = \hat{X}_n^{-n} \hat{X}_{n-1}^{-k} - \hat{X}_n^{-k} \hat{X}_n^{-(n-1)} = \alpha \mathbb{1}$$

for some $k = 1, \ldots, n-1$, commutation relations with P^0 give $\hat{Y}_{n-1}^{-k} = 0$. The simplicity of the Lie algebra C_{n-1} generated by the $\hat{Y}$'s leads to $\hat{Y}_j{}^i = 0$ for all $i, j = \pm 1, \ldots, \pm (n-1)$ so that condition (3.2) is fulfilled and Lemma and Corollary 1 can be applied.

So in all cases admissible by possibility (a) the realization τ is a Schur realization and proof is completed.

4. *Conclusion*

Denote by $n_{\min}$ the minimal number of canonical pairs such that nontrivial realization of a given Lie algebra exists in $D_{2n_{\min}}$. The values for the four series of complex simple Lie algebras are given as follows (Joseph, 1974):

	A_n	$B_n, n > 1$	C_n	$D_n, n > 2$
$n_{\min}$	n	$2n-2$	n	$2n-3$

Denote further by $k_{\max}$ such a maximal integer that all realizations in $W_{2(n_{\min}+k_{\max})}$ of a given Lie algebra are Schur realizations. For classical

Lie algebras $k_{\max}$ exists and is given as follows:

	A_n	$B_n, n > 1$	C_n	$D_n, n > 2$
$k_{\max}$	0	1	$n-1$	1

As to $k_{\max}$ for A_n see (Simoni and Zaccaria, 1969; Joseph, 1974; Havlíček and Exner, 1975) for B_n and D_n; equality $k_{\max} = n - 1$ for the Lie algebras C_n is proved just in the present note. Maximality of $k_{\max}$ follows from the existence of one-parameter sets of realizations in $W_{2(n_{\min} + k_{\max})}$ with Casimir operators depending on this parameter (Simoni and Zaccaria, 1969; Joseph, 1974; Havlíček and Lassner, 1976. b; Havlíček and Exner, 1975): Substituting this parameter by $q_{n_{\min} + k_{\max} + 1}$ from the new pair $q_{n_{\min} + k_{\max} + 1}, p_{n_{\min} + k_{\max} + 1}$ we obtain non-Schur-realizations in $W_{2(n_{\min} + k_{\max} + 1)}$. The second set of values above shows the remarkable distinction, as to $k_{\max}$, between C_n and the other classical Lie algebras; realizations of C_n in $D_{2(2n-2)}$, however, remain still related, in the sense of Definition 1, to the standard minimal one.

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